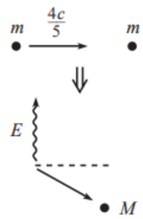
Physics IV
ISI B.Math
Final Exam : April 26, 2023

Total Marks: 50
Time : 3 hours
Answer all questions

1. (**Marks : 7 + 3 = 10**)



(a) A mass m moving at speed $\frac{4}{5}c$ collides with another mass m at rest. The collision produces a photon with energy E traveling perpendicular to the original direction, and a mass M traveling in another direction as shown in the figure. In terms of E and m,what is M? What is the largest value of E (in terms of m) for which this setup is possible ?

(b) Write down a relativistic expression for the kinetic energy of a particle of mass m and speed v. Show that it reduces to the usual Newtonian expression in the appropriate limit.

2. (**Marks : 3 + 3 + 4 + 2 = 12**)

The annihilation operator for the harmonic oscillator in 1-d is defined as $\hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega_0}\right)$ where m is the mass and ω_0 is the angular frequency of the harmonic oscillator

(a) If $\hat{H}$ is the Hamiltonian for the above one dimensional harmonic oscillator and $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator, show that $\hat{N}$ and $\hat{H}$ have simultaneous eigenstates. Find the eigenvalues of the energy in terms of the eigenvalues n of the number operator where $n = 0, 1, 2\cdots$

(b) Given that $|n>$ is a normalized eigenstate corresponding to the eigenvalue n of the Hamiltonian of the harmonic oscillator where n is an integer and $\hat{a}|n> = \sqrt{n}|n-1>$ and $\hat{a}^\dagger|n> = \sqrt{n+1}|n+1>$, Find $<\hat{x^3}>$ for the nth energy eigenstate of the harmonic oscillator.

(c) Show that the average kinetic energy $<T>$ is equal to the average potential energy $<V>$ for any energy eigenstate of the harmonic oscillator

(d) In the case of the 1-d harmonic oscillator the energy levels are non-degenerate, i.e, an energy eigenvalue is associated with a unique eigenstate. Now if we generalize to three dimensions and consider the potentials i) $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$ and (ii)$\frac{1}{2}m({\omega_1}^2x^2 + {\omega_2}^2y^2 + {\omega_3}^2z^2)$ of the three dimensional isotropic and anisotropic harmonic oscillators, discuss whether the feature of nondegeneracy persists. You do not have to solve the entire problem in three dimensions all over again, but just have to give a plausible argument.

3. (**Marks : 3 + 3 + 4 = 10**)

Consider the one dimensional normalized wave functions ψ_0 and ψ_1 with the following properties $\psi_0(x) = \psi_0(-x) = \psi_0^*(x)$ and $\psi_1(x) = N\frac{d\psi_0}{dx}$
Consider also the linear combination $\psi(x) = c_1\psi_0(x) + c_2\psi_1(x)$ where $|c_1|^2 + |c_2|^2 = 1$. The constants N, c_1, c_2 are considered to be known.

(a) Show that ψ_0 and ψ_1 are orthogonal and ψ is normalized.

(b) Compute the expectation values of x and p in the states ψ_0, ψ_1 and ψ.

(c) Compute the expectation value of the kinetic energy in the state ψ_0 and demonstrate that

$$< \psi_0|T^2|\psi_0 >=< \psi_0|T|\psi_0 >< \psi_1|T|\psi_1 >$$

and

$$< \psi_1|T|\psi_1 >\geq< \psi|T|\psi >\geq< \psi_0|T|\psi_0 >$$

4. (**Marks: 10**)

The wave function of a free particle at $t = 0$ is given by

$$\psi(x, 0) = \frac{1}{\sqrt{a}} \text{ for } -\frac{a}{2} \leq x \leq \frac{a}{2}$$

$$\text{and } \psi(\text{x}, 0) = 0 \text{ elsewhere.}$$

If the momentum of the particle is measured at $t = 0$, show that the probability $P(k)$ of finding the momentum of the particle between $\hbar k$ and $\hbar(k + dk)$ is given by

$$P(k) = \frac{2}{\pi a}\frac{\sin^2(ka/2)}{k^2}$$

What is the most probable value of the momentum that will be found on measurement? Which values of momentum will never be found ?
(Hint: Recall that the eigenstates of momentum $\hat{p}$ with eigenvalue $\hbar k$ are given by $\frac{1}{\sqrt{2\pi}}e^{ikx}$ and expand the wavefunction as a superposition of these eigenstates)

5. (**Marks: 5 + 3 = 8**)

(a) For any observable A that does not explicitly depend on time, show that

$$\frac{d < A >}{dt} = \frac{i}{\hbar} < [H, A] >$$

where H is the Hamiltonian of the system. Under what conditions will it be possible to measure H and A consecutively with no uncertainty in the values obtained ?

(b) The Hamiltonian $H = \frac{p^2}{2m} + V(x)$ Where $V(x)$ is the potential energy of the system. Using the result in (a) show that

$$\frac{d < p >}{dt} =< -\frac{dV}{dx} >$$